Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 1 Version 1.0 January 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Characterizations of Space-Like Rectifying Curves in the Minkowski Space-Time

By Ahmad T. Ali, Mehmet Ä Onder
Celal Bayar University Muradiye Campus, Manisa, TURKEY

Abstract - In this work, a space-like rectifying curve with space-like principal normal in the Minkowski space-time E_1^4 is defined as a curve whose position vector always lies in orthogonal complement $N^{\perp}$ of its principal normal vector field N. Also, we characterized such curves in terms of their curvature functions and we obtained the necessary and sficient conditions for such curve to be a rectifying curve.

Keywords : *Space-like rectifying curve; Frenet equations; Minkowski space-time.*

GJSFR Classification MSC: *14H45; 14H50; 53C40; 53C50*

SOME CHARACTERIZATIONS OF SPACE-LIKE RECTIFYING CURVES IN THE MINKOWSKI SPACE-TIME

Strictly as per the compliance and regulations of :

Some Characterizations of Space-Like Rectifying Curves in the Minkowski Space-Time

Ahmad T. Ali [α], Mehmet Ä Onder [Ω]

Abstract - In this work, a space-like rectifying curve with space-like principal normal in the Minkowski space-time E_1^4 is defined as a curve whose position vector always lies in orthogonal complement $N^{\perp}$ of its principal normal vector field N. Also, we characterized such curves in terms of their curvature functions and we obtained the necessary and sficient conditions for such curve to be a rectifying curve.

Keywords : Space-like rectifying curve; Frenet equations; Minkowski space-time.

I. Introduction

Lorentzian geometry helps to bridge the gab between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. The fact that relativity theory is expressed in terms of Lorentzian geometry is attractive for geometers, who can penetrate surprising quickly into cosmology (redshift, expanding universe and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light and black holes) [18]. Despite its long history, the theory of curve is still one of the most important interesting topics in a differential geometry and its is being study by many mathematicians until now, see for example [1, 2, 3, 4, 5, 16, 19, 21, 24].

In the Euclidean space E^3, rectifying curves are introduced by Chen in [7] as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields $\vec{T}$ and $\vec{B}$ of the curve. Therefore, the position vector $\vec{\alpha}$ of a rectifying curve satisfies the equation

$$\vec{\alpha}(s)=\lambda(s)\vec{T}(s)+\mu(s)\vec{B}(s),$$

for some differentiable functions λ and μ in arclength function s . The Euclidean rectifying curves are studied in [7, 8]. In particular, it is shown in [8] that there exists a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential geometry in defining the curves of constant precession. The rectifying curves are also studied in [8] as the extremal curves. In the Minkowski 3-space E_1^3, the rectifying curves are investigated in [10]. The rectifying curves are also studied in [11] as the centrodes and extremal curves. In the Euclidean 4-space E^4, the rectifying curves are investigated in [9].

In analogy with the rectifying curve, the curve whose position vector always lies in its normal plane spanned by the principal normal and the binormal vector fields $\vec{N}$ and $\vec{B}$ of the curve is called normal curve in Euclidean 3-space E^3 and it is well known that normal curves are spherical curves in E^3 [8]. Similar definition and characterizations of space-like, time-like (and also null) and dual time-like normal curves are given in references [10, 11, 17]. The space-like normal curve in Minkowski 4-space E_1^4 is defined in [14] as a curve whose position vector always lies in the orthogonal complement $\vec{T}^{\perp}$ of its tangent vector field $\vec{T}$ which is given by

$$\vec{T}^{\perp}=\{\vec{W}\in E_1^4 \mid g(\vec{W},\vec{T})=0\}.$$

In [6], Camci and others have shown that a space-like curve lies in pseudohyperbolic space H_0^3 iff the following equation holds

$$\vec{\alpha}-m=-(1/k_1)\vec{N}-(1/k_2)(1/k_1)'\vec{B}_1+(1/k_3)[k_2/k_1+((1/k_2)(1/k_1)')']\vec{B}_2,$$

Where m is constant, k_1, k_2 and k_3 are the first, the second and the third curvatures of the curve α , respectively. By using the definition of space-like normal curves in Minkowski 4 - space E_1^4 and the last equality, it follows that every space-like curve lying in pseudohy-perbolic space H_0^3 is a normal curve in Minkowski 4-space.

In this paper, in analogy with the Minkowski 3-dimensional case, we define the rectifying curve in the Minkowski 4-space E_1^4 as a curve whose position vector always lies in the orthogonal complement $\vec{N}^{\perp}$ of its principal normal vector field $\vec{N}$ Consequently, $\vec{N}^{\perp}$ is given by

$$\vec{N}^{\perp}=\{\vec{W}\in E_1^4 \mid g(\vec{W},\vec{N})=0\},$$

Author α : King Abdul Aziz University, Faculty of Science, Department of Mathematics, PO Box 80203, Jeddah, 21589, Saudi Arabia. Al-Azhar University, Faculty of Science, Mathematics Department, Nasr City, 11448, Cairo, EGYPT. E-mail : atali71@yahoo.com
Author Ω : Department of Mathematics Faculty of Science and Arts, Celal Bayar University Muradiye Campus, Manisa, TURKEY. E-mail : mehmet.onder@bayar.edu.tr

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

Where $g(.,.)$ denotes the standard pseudo scalar product in E_1^4. Hence $\vec{N}^{\perp}$ is a 3-dimensional subspace of E_1^4, spanned by the tangent, the first binormal and the second binormal vector fields $\vec{T}$, $\vec{B}_1$ and $\vec{B}_2$ respectively. Therefore, the position vector with respect to some chosen origin, of a space-like rectifying curve $\vec{\alpha}$ in Minkowski space-time E_1^4, satisfies the equation

$$\vec{\alpha}(s) = \lambda(s)\vec{T}(s) + \mu(s)\vec{B}_1(s) + \nu(s)\vec{B}_2, \tag{1}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s. Next, characterize space-like rectifying curves in terms of their curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ and give the necessary and the sufficient conditions for arbitrary curve in E_1^4 to be a rectifying. Moreover, we obtain an explicit equation of a space-like rectifying curve in E_1^4 and give the relation between rectifying and normal space-like curves in E_1^4.

II. Preliminaries

In this section, we prepare basic notations on Minkowski space-time E_1^4. Let $\vec{\alpha} : I \subset R \to E_1^4$ be arbitrary curve in the Minkowski space - time E_1^4. Recall that the curve $\vec{\alpha}$ is said to be unit speed (or parameterized by arclength function s) if $g(\vec{\alpha}', \vec{\alpha}') = \pm 1$, where $g(.,.)$ denotes the standard pseudo scalar product in E_1^4 given by

$$g(\vec{v}, \vec{w}) = -v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4,$$

for each $\vec{v} = (v_1, v_2, v_3, v_4) \in E_1^4$ and $\vec{w} = (w_1, w_2, w_3, w_4) \in E_1^4$. An arbitrary vector $\vec{v} \in E_1^4$ can have one of three Lorentzian causal characters; it can be space - like if $g(\vec{v}, \vec{v}) > 0$ or $\vec{v} = 0$, time - like if and null (light-like) if $g(\vec{v}, \vec{v}) = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null (light-like). Also recall that the pseudo-norm of an arbitrary vector $\vec{v} \in E_1^4$ is given by $\| \vec{v} \| = \sqrt{|g(\vec{v}, \vec{v})|}$. Therefore $\vec{v}$ is a unit is a unit vector if $g(\vec{v}, \vec{v}) = \pm 1$. The velocity of the curve $\vec{\alpha}(s)$ is given by $\| \vec{\alpha}'(s) \|$. Next, vectors $\vec{v}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(\vec{v}, \vec{w}) = 0$.

Denote by $\{\vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s)\}$ the moving Frenet frame along the curve $\vec{\alpha}(s)$ in the space E_1^4, where $\vec{T}(s)$, $\vec{N}(s)$, $\vec{B}_1(s)$ and $\vec{B}_2(s)$ are the tangent, principal normal, the first binormal and second binormal fields, respectively. For an arbitrary space - like curve $\vec{\alpha}(s)$ with space - like principal normal $\vec{N}$ in the space E_1^4, the following Frenet formula are given in [23, 20, 6, 15, 22, 25]:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\varepsilon\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & \kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \tag{2}$$

Where

$$g(\vec{B}_1, \vec{B}_1) = -g(\vec{B}_2, \vec{B}_2) = \varepsilon = \pm 1, \quad g(\vec{T}, \vec{T}) = g(\vec{N}, \vec{N}) = 1. \tag{3}$$

Recall the functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are called respectively, the first, the second and the third curvatures of space-like curve $\vec{\alpha}(s)$. Here, ε determines the kind of space - like curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a space-like curve with space - like first binormal $\vec{B}_1$ and time - like second binormal B_2. If $\varepsilon = -1$, then $\alpha(s)$ is a space-like curve with time - like first binormal $\vec{B}_1$ and space - like second binormal B_2. If $\kappa_3(s) \neq 0$ for each $s \in I \subset R$, the curve $\vec{\alpha}$ lies fully in E_1^4. Recall that the pseudohyperbolic space $H_0^3(1)$ in E_1^4, centered at the origin, is the hyperquadric defined by

$$H_0^3(1) = \{\vec{X} \in E_1^4 \mid g(\vec{X}, \vec{X}) = -1\}.$$

Recently, Yilmaz et al. [27, 26] defined a vector product in Minkowski spacetime $\mathbf{E}_1^4$ as follows:

Definition 2.1. *Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $\boldsymbol{E}_1^4$. The vector product in Minkowski space-time E_1^4 is defined by the determinant*

$$a \wedge b \wedge c = -\begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}, \tag{4}$$

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012
58

where e_1, e_2, e_3 *and* e_4 *are mutually orthogonal vectors (coordinate direction vectors) satisfying equations*

$$e_1 \wedge e_2 \wedge e_3 = e_4, \quad e_2 \wedge e_3 \wedge e_4 = e_1, \quad e_3 \wedge e_4 \wedge e_1 = e_2, e_4 \wedge e_1 \wedge e_2 = -e_3.$$

Lemma 2.2. *Let* $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ *and* $c = (c_1, c_2, c_3, c_4)$ *be vectors in* $\boldsymbol{E}_1^4$. *From the definition of vector product, there is a property in Minkowski space - time* E_1^4 *as the following:*

$$g(a \wedge b \wedge c, a) = g(a \wedge b \wedge c, b) = g(a \wedge b \wedge c, c) = 0. \tag{5}$$

The proof of above lemma is elementary.

III. Some Characterizations of Rectifying Curves In E_1^4

In this section, we firstly characterize the space-like rectifying curves with space-like principal normal in Minkowski space - time in terms of their curvatures. Let $\vec{\alpha} = \vec{\alpha}(s)$ be a unit speed space - like rectifying curve in E_1^4, with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ By definition, the position vector of the curve $\vec{\alpha}$ sacrifies the equation (1), for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Differentiating the equation (1) with respect to s and using the Frenet equations (2), we obtain

$$\vec{T} = \lambda' \vec{T} + (\lambda\kappa_1 - \varepsilon\mu\kappa_2)\vec{N} + (\mu' + \nu\kappa_3)\vec{B}_1 + (\nu' + \mu\kappa_3)\vec{B}_2. \tag{6}$$

It follows that

$$\begin{aligned} &\lambda' = 1, \\ &\lambda\kappa_1 - \varepsilon\mu\kappa_2 = 0, \\ &\mu' + \nu\kappa_3 = 0, \\ &\nu' + \mu\kappa_3 = 0, \end{aligned} \tag{7}$$

and therefore

$$\begin{aligned} &\lambda = s + c, \\ &\mu = \varepsilon \frac{\kappa_1(s)(s+c)}{\kappa_2}, \\ &\nu = -\varepsilon \frac{\kappa_1(s)\kappa_2(s) + (s+c)(\kappa_1'(s)\kappa_2(s) - \kappa_1(s)\kappa_2'(s))}{\kappa_2^2(s)\kappa_3(s)}, \end{aligned} \tag{8}$$

where $c \in R$. In this way the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ of the curve $\alpha(s)$. Moreover, by using the last equation in (7) and relation (8) we easily find that the curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ satisfy the equation

$$\frac{\kappa_1(s)\kappa_3(s)(s+c)}{\kappa_2(s)} - \Big[\frac{\kappa_1(s)\kappa_2(s) + (s+c)(\kappa_1'(s)\kappa_2(s) - \kappa_1(s)\kappa_2'(s))}{\kappa_2^2(s)\kappa_3(s)}\Big]' = 0, \quad c \in R. \tag{9}$$

The condition (9) can be written as:

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} - \frac{1}{\kappa_3(s)}\frac{d}{ds}\Big[\frac{1}{\kappa_3(s)}\frac{d}{ds}\Big(\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\Big)\Big] = 0. \tag{10}$$

If we change the variable s by the variable t as the following

$$\frac{d}{dt} = \frac{1}{\kappa_3(s)}\frac{d}{ds} \quad \text{or} \quad t = \int_0^s \kappa_3(s)ds,$$

the equation (10) takes the following form

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} - \frac{d^2}{dt^2}\Big[\frac{\kappa_1(s)(s+c)}{\kappa_2(s)}\Big] = 0. \tag{11}$$

General solution of this equation is

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} = \varepsilon\Big(A\cosh \textstyle\int_0^s \kappa_3(s)ds + B\sinh \int_0^s \kappa_3(s)ds\Big), \tag{12}$$

where A and B are arbitrary constants. Then from (8) we have

$$\begin{aligned}\lambda(s) &= s + c\\ \mu(s) &= A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\\ \nu(s) &= -\Big(A\sinh\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\Big).\end{aligned} \tag{13}$$

Conversely, assume that the curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ of an arbitrary unit speed space - like curve in E_1^4, satisfy the equation (12). Let us consider the vector $\vec{X} \in E_1^4$ given by

$$\begin{aligned}\vec{X}(s) &= \vec{\alpha}(s) - (s+c)\vec{T}(s) - \Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big)\vec{B}_1(s)\\ &+\Big(A\sinh\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\Big)\vec{B}_2(s).\end{aligned} \tag{14}$$

By using the relations (2) and (12), we easily find $\vec{X}'(s) = 0$, which means that $\vec{X}$ is a constant vector. This implies that $\alpha(s)$ is congruent to a rectifying curve. In this way, the following theorem is proved.

Theorem 3.1. *Let $\vec{\alpha}(s)$ be unit speed space - like curve with space - like principal normal in E_1^4 and with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. Then $\vec{\alpha}(s)$ is congruent to a space - like rectifying curve if and only if*

$$\frac{\kappa_1(s)(s+c)}{\kappa_2(s)} = \varepsilon\Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big).$$

In particular, assume that all curvature functions $\kappa_1(s)$, $\kappa_2(s)$, and $\kappa_3(s)$ of space - like rectifying curve, $\vec{\alpha}$ in E_1^4 are constant and different from zero. Then equation (9) easily implies a contradiction. Hence we obtain the following theorem.

Theorem 3.2. *There are no space - like rectifying curves with space - like principal normal lying in E_1^4, with non-zero constant curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$.*

In the next theorem, we give the necessary and the suficient conditions for the space - like curve $\alpha(s)$ in E_1^4 to be a rectifying curve.

Theorem 3.3. *Let $\alpha(s)$ be unit speed space-like rectifying curve with space-like prini-pal normal in E_1^4, with non-zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. Then the following statements hold:*

(i) The distance function $\rho(s) = \| \vec{\alpha}(s) \|$ satisfies $\rho^2(s) = s^2 + c_1 s + c_2$, $c_1 \in R$ and $c_2 \in R_0$.

(ii) The tangential component of the position vector of the space-like rectifying curve is given by $g(\vec{\alpha}(s), \vec{T}(s)) = s + c$, $c \in R$.

(iii) The normal component $\vec{\alpha}^N(s)$ of the position vector of the space - like rectifying curve has constant length and the distance function $\rho(s)$ is non - constant.

(iv) The first binormal component and the second binormal component of the position vector of the space-like rectifying curve are respectively given by

$$\begin{aligned}g(\vec{\alpha}(s), \vec{B}_1(s)) &= \varepsilon\Big(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\Big)\\ g(\vec{\alpha}(s), \vec{B}_2(s)) &= \varepsilon\Big(A\sinh\int_0^s \kappa_3(s)ds - B\cosh\int_0^s \kappa_3(s)ds\Big).\end{aligned} \tag{15}$$

Conversely, if $\vec{\alpha}(s)$ is a unit speed curve in E_1^4 with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ and one of the statements (i), (ii), (iii) or (iv) holds, then $\vec{\alpha}(s)$ is a space - like rectifying curve.

Proof. Let us first suppose that $\vec{\alpha}(s)$ is a unit speed space - like rectifying curve in E_1^4 with non - zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. The position vector of the curve $\vec{\alpha}(s)$ satisfies the equation (1), where the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ satisfy relation (13). From relation (1) and (13) we have

$$\begin{aligned}g(\vec{\alpha}, \vec{\alpha}) &= \lambda^2 + \varepsilon\Big(\mu^2(s) - \nu^2(s)\Big),\\ &= (s+c)^2 + \varepsilon(A^2 - B^2).\end{aligned} \tag{16}$$

Therefore, $\rho^2(s) = s^2 + c_1 s + c_2$, $c_1 \in R$ and $c_2 \in R_0$, which proves statement (i).

But using the relations (1) and (8) we easily get $g(\vec{\alpha}(s), \vec{T}(s)) = s + c$, $c \in R$, so the statement (ii) is proved.

Note that the position vector of an arbitrary curve $\vec{\alpha}(s)$ in E_1^4 can be decomposed as $\vec{\alpha}(s) = m(s)\vec{T}(s) + \vec{\alpha}^N(s)$, where $m(s)$ is arbitrary differentiable function and $\vec{\alpha}^N(s)$ is the normal component of the position vector. If $\vec{\alpha}(s)$ is a space-like rectifying curve, relation (1) implies $\vec{\alpha}^N(s) = \mu(s)\vec{B}_1(s) + \nu(s)\vec{B}_2(s)$ and therefore $g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = \varepsilon\left(\mu^2(s) - \nu^2(s)\right)$.Moreover, by using (13), we find $\| \vec{\alpha}^N(s) \| = \varepsilon(A^2 - B^2) = a$, $a \in R$. By statement (i), $\rho(s)$ is non-constant function, which proves statement (iii).

Finally, using (1), (3) and (13) we easily obtain (15), which proves statement (iv).

Conversely, assume that statement (i) holds. Then $g(\vec{\alpha}(s), \vec{\alpha}(s)) = s^2 + c_1 s + c_2,\ c_1 \in R,\ c_2 \in R$ By differentiating the previous equation two times with respect to s and using (2), we obtain $g(\vec{\alpha}(s), \vec{N}(s)) = 0$, which implies that $\vec{\alpha}$ is a space-like rectifying curve.

If statement (ii) holds, in a similar way it follows that $\vec{\alpha}$ is a space-like rectifying curve.

If statement (iii) holds, let us put $\vec{\alpha}(s) = m(s)\vec{T}(s) + \vec{\alpha}^N(s)$, where $m(s)$ is arbitrary differentiable function. Then

$$g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = g(\vec{\alpha}(s), \vec{\alpha}(s)) - 2g(\vec{\alpha}(s), \vec{T}(s))m(s) + m^2(s). \tag{17}$$

Since $g(\vec{\alpha}(s), \vec{T}(s)) = m(s)$, it follows that

$$g(\vec{\alpha}^N(s), \vec{\alpha}^N(s)) = g(\vec{\alpha}(s), \vec{\alpha}(s)) - g(\vec{\alpha}(s), \vec{T}(s))^2, \tag{18}$$

where $g(\vec{\alpha}(s), \vec{\alpha}(s)) = \rho^2(s) \neq$ constant. Defferentiating the previous equation with respect to s and using (2), we find

$$\kappa_1(s)\, g(\vec{\alpha}(s), \vec{T}(s))\, g(\vec{\alpha}(s), \vec{N}(s)) = 0. \tag{19}$$

It follows that $g(\vec{\alpha}(s), \vec{N}(s)) = 0$ and hence the space-like curve $\vec{\alpha}$ is a rectifying.

If the statement (iv) holds, by taking the derivative of the equation

$$g(\vec{\alpha}(s), \vec{B}_1(s)) = \varepsilon\left(A\cosh\int_0^s \kappa_3(s)ds + B\sinh\int_0^s \kappa_3(s)ds\right), \tag{20}$$

with respect to s and using (2), we obtain

$$-\varepsilon\kappa_2(s)g(\vec{\alpha}(s), \vec{N}(s)) + \kappa_3 g(\vec{\alpha}(s), \vec{B}_2(s)) = \varepsilon\kappa_3\left(A\sinh\int_0^s \kappa_3(s)ds + B\cosh\int_0^s \kappa_3(s)ds\right). \tag{21}$$

By using (15), the last equation becomes $g(\vec{\alpha}(s), \vec{N}(s)) = 0$, which means that $\vec{\alpha}$ is a space-like rectifying curve. This proves the theorem.

In the next theorem, we find the parametric equation of a rectifying curve.

Theorem 3.4. *Let $\alpha : I \subset R \to E_1^4$ be a space-like curve with space-like principal normal in E_1^4 given by $\vec{\alpha}(t) = \rho(t)\ \vec{y}(t)$ where $\rho(t)$ is an arbitrary positive function and $\vec{y}(t)$ is a unit speed space-like curve lying in pseudohyperbolic space $H_0^3(1)$. Then $\vec{\alpha}$ is a space-like rectifying curve if and only if*

$$\rho(t) = \frac{a}{\cosh(t + t_0)}, \quad a \in R_0, \quad t_0 \in R. \tag{22}$$

Proof. Let $\vec{\alpha}$ be a curve in E_1^4 given by

$$\vec{\alpha}(t) = \rho(t)\vec{y}(t) \tag{23}$$

where $\rho(t)$ is arbitrary positive function and $\vec{y}(t)$ is a unit speed space-like curve in the pseudohyperbolic space $H_0^3(1)$.. By taking the derivative of the previous equation with respect to t, we get

$$\vec{\alpha}'(t) = \rho'(t)\vec{y}(t) + \rho(t)\vec{y}'(t). \tag{24}$$

Hence the unit tangent vector of $\vec{\alpha}$ is given by

$$\vec{T} = \frac{\rho'(t)}{v(t)}\vec{y}(t) + \frac{\rho(t)}{v(t)}\vec{y}'(t), \tag{25}$$

where $v(t) = \| \vec{\alpha}'(t) \|$ is the speed of $\vec{\alpha}$. Differentiating the equation (25) with respect to t, we find

$$\vec{T}' = \left(\frac{\rho'}{v}\right)'\vec{y} + \left(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho + \rho'')}{v^3}\right)\vec{y}' + \left(\frac{\rho}{v}\right)\vec{y}''. \tag{26}$$

Let $\vec{Y}$ be the unit vector field in E_1^4 satisfying the equations $g(\vec{Y},\vec{y}) = g(\vec{Y},\vec{y}') = 0$. Then $\{\vec{y},\vec{y}',\vec{Y},\vec{y}\wedge\vec{y}' \wedge \vec{Y}\}$ is orthonormal frame in E_1^4. Therefore decomposition of $\vec{y}''$ with respect the frame $\{\vec{y},\vec{y}',\vec{Y},\vec{y}\wedge\vec{y}'\wedge\vec{Y}\}$ reads

$$\begin{aligned}\vec{y}'' = \; & g(\vec{y}'',\vec{y})g(\vec{y},\vec{y})\,\vec{y} + g(\vec{y}'',\vec{y}')\,g(\vec{y}',\vec{y}')\vec{y}' + g(\vec{y}'',\vec{Y})\,g(\vec{Y},\vec{Y})\vec{Y} \\ & + g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})\,g(\vec{y}\wedge\vec{y}'\wedge\vec{Y},\vec{y}\wedge\vec{y}'\wedge\vec{Y})\vec{y}\wedge\vec{y}'\wedge\vec{Y}.\end{aligned} \tag{27}$$

Since $g(\vec{y},\vec{y}) = -1$ and $g(\vec{y}',\vec{y}') = 1$, it follows that $g(\vec{y}'',\vec{y}) = -1$ and $g(\vec{y}'',\vec{y}') = 0$, so the equation (27) the equation (27) becomes

$$\vec{y}'' = \vec{y} + g(\vec{y}'',\vec{Y})\,\vec{Y} + g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})\,\vec{y}\wedge\vec{y}'\wedge\vec{Y}. \tag{28}$$

Substituting (28) into (26) and applying Frenet formulas for arbitrary speed curves in E_1^4, we find

$$\begin{aligned}\kappa_1 v\vec{N} \; & = \Big[\Big(\frac{\rho'}{v}\Big)' + \frac{\rho}{v}\Big]\vec{y} + \Big(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho+\rho'')}{v^3}\Big)\vec{y}' + \Big(\frac{\rho}{v}\Big)g(\vec{y}'',\vec{Y})\vec{Y} \\ & + \frac{g(\vec{y}'',\vec{y}\wedge\vec{y}'\wedge\vec{Y})}{v}\,\vec{\alpha}\wedge\vec{y}'\wedge\vec{Y}.\end{aligned} \tag{29}$$

Since $g(\vec{y},\vec{y}) = -1$, we have $g(\vec{y},\vec{y}') = 0$ and thus $g(\vec{\alpha},\vec{y}') = 0$. We also have $g(\vec{\alpha},\vec{Y}) = 0$. By definition, $\vec{\alpha}$ is a space - like rectifying curve in E_1^4 if and only if $g(\vec{\alpha},\vec{N}) = 0$. Therefore, after taking the scalar product of (29) with $\vec{\alpha}$, we have $g(\vec{\alpha},\vec{N}) = 0$ if and only if

$$\Big(\frac{\rho'}{v}\Big)' + \frac{\rho}{v} = 0, \tag{30}$$

whose general solutions are $\rho(t) = \dfrac{a}{\cosh(t+t_0)}$ or $\rho(t) = \dfrac{a}{\sinh(t+t_0)}$, $a \in R_0, t \in R$. Since, $g(\vec{T},\vec{T}) = 1$, it follows that $\rho(t) = \dfrac{a}{\cosh(t+t_0)}$ is the only solution. This proves the theorem.

In Theorem 3.4, since $\vec{y}(s)$ is a unit speed space - like curve in the pseudohyperbolic space $H_0^3(1)$, $\vec{y}(s)$ is a space - like normal curve in Minkowski 4- space E_1^4. So, Theorem 3.4 gives the relation between space - like rectifying and space-like normal curves in Minkowski 4-space E_1^4. Then we can give the following corollary.

Corollary 3.5. *In Minkowski 4-space E_1^4, the construction of the space - like rectifying curve with space - like principal normal can be made by using the space - like normal curves.*

Example: Let us consider the curve

$$\vec{\alpha}(s) = \frac{a}{\cosh(s+s_0)}\Big(\sqrt{2}\cosh(s/\sqrt{3}), \sqrt{2}\sinh(s/\sqrt{3}), \sin(s/\sqrt{3}), \cos(s/\sqrt{3})\Big),$$

where $a \in R_0, s_0 \in R$ in E_1^4 This curve has a form $\vec{\alpha}(s) = \rho(s)\vec{y}(s)$ where $\rho(s) = \dfrac{a}{\cosh(s+s_0)}$ and $\vec{y}(s) = \Big(\sqrt{2}\cosh(s/\sqrt{3}), \sqrt{2}\sinh(s/\sqrt{3}), \sin(s/\sqrt{3}), \cos(s/\sqrt{3})\Big)$ Since $g(\vec{y}(s),\vec{y}(s)) = -1$ and $\| \vec{y}'(s) \| = 1$, $\vec{y}(s)$ is a unit speed space-like curve in pseudohyper- bolic space $H_0^3(1)$. According to theorem 3.4, $\vec{\alpha}$ is a space - like rectifying curve lying fully in E_1^4.

References Références Referencias

1. A.T. Ali, *Position vectors of spacelike helices from intrinisic equations in Minkowski 3-space*, Nonl. Anal. Theory Meth. Appl. 2010, 73, 1118 -1126.
2. A.T. Ali and R. Lopez, *Timelike* B2*-slant helices in Minkowski* E41 , Archivum. Math. (Brno) 2010, 64, 39-46.
3. A.T. Ali and R. Lopez, *Slant helices in Minkowski space* E31 , J. Korean Math. Soc. 2011, 48, 159-167.
4. A.T. Ali and M. Turgut, *Position vector of a timelike slant helix in Minkowski 3-space*, J. Math. Anal. Appl. 2010, 365, 559-569.
5. A.T. Ali and M. Turgut, *Determination of time like general helices in Minkowski 3-space*, Preprint 2009: arXiv:0906.3851v1 [math.DG].
6. C. Camci, K, İlarslan, E.Šućurović, *On pseudohyperbolical curves in Minkowski space-time*, Turk. J. Math., 2003, 27: 315-328.

7. B.Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly, 2003, 110: 147-152.
8. B.Y. Chen and F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Acadimia Sinica, 2005, 2: 77-90.
9. K. İlarslan and E. Nešović, *Some characterizations of rectifying curves in the Eu-clidean space E*4, Turk. J. Math. 2008, 32: 21-30.
10. K. İlarslan, E. Nešović and M. Petrovié -Torgašev, *Some characterizations of rectifying curves in Minkowski 3-space*, Novi. Sad. J. Math. 2003, 33(2): 23-32.
11. K. İlarslan and E. Nešović, *On rectifying curves as centrodes and extremal curves in the Minkowski 3-space*, Novi. Sad. J. Math. 2007, 37(1): 53-64.
12. K. İlarslan, *Spacelike Normal Curves in Minkowski space E*31 , Turk. J. Math. 2005, 29: 53-63.
13. K. İlarslan and E. Nešović, *Timelike and null Normal Curves in Minkowski space*, Indian J. Pure and Appl. Math., 2004, 35(7): 881-888.
14. K. İlarslan and E. Nešović, *Spacelike and timelike normal curves in Minkowski space*. to appear in Publ. Inst. Math. (Beograd), Vol. 85(99), (2009).
15. M. Kazaz, M. Ä Onder and H. Kocayigit. *Spacelike Curves of constant Breadth in Minkowski 4-space*, Int. J. Math. Anal. 2008, 2(22): 1061-1068.
16. L. Kula, N. Ekmekci, Y. Yayliand and K. Ilarslan. *Characterizations of slant helices in Euclidean 3-space*, Tur. J. Math. 2009, 33, 1-13.
17. M. Önder, *Dual time like normal and dual time like spherical curves in dual Minkowski space D*31, SDU Journal of Science 2006, 1(1-2): 77-86.
18. B. O'Neill. Semi-Riemannian Geometry with Application to relativity, Academic Press, New York, 1983.
19. M. Ozdemir and A.C. Coken, *Backlund transformation for non-lightlike curves in Minkowski 3-space*, Chaos, Solitons and Fractals 2009, 42, 2540-2545.
20. M. Petrović-Torga·sev and E. ·Sucurovic, *W-Curves in Minkowski space-time*, Novi. Sad. J. Math. 2002, 32(2): 55-65.
21. M. Sakaki, *Notes on null curves in Minkowski spaces*, Tur. J. Math. 2009, 33, 1-8.
22. M. Turgut and S. Yilmaz, *On the Frenet frame and a characterization of space-like involute-evolute curve couple in Minkowski space-time*, Int. Math. Forum. 2008, 3(16):793-801.
23. J. Walrave, Curves and surfaces in Minkowski space. doctoral thesis, K U Leuven, Fac Sci, Leuven, 1995.
24. S. Yilmaz and M. Turgut, *A new version of Bishop frame and an application to spherical images*, J. Math. Anal. Appl. 2010, doi:10.1016/j.jmaa.2010.06.012.
25. S. Yilmaz and M. Turgut, *Relations among Frenet apparatus of space-like Bertrand W-curve couples in Minkowski space-time*, Int. Math. Forum. 2008, 3(32): 1575-1580.
26. S. Yilmaz and M. Turgut, *On the Di®erential Geometry of the curves in Minkowski space-time I*, Int. J. Contemp. Math. Sci. 2008, 3(27): 1343-1349.
27. S. Yilmaz, E. Ozyilmaz, M. Turgut, *On the Di®erential Geometry of the curves in Minkowski space-time II*, Int. J. Comput. Math. Sci. 2009, 3(2), 53-55.

This page is intentionally left blank